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Vibration frequencies of elastic beams with extra point masses

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ABSTRACT

Free vibration responses of elastic beams with extra point masses are under investigations by dint of the scaled boundary finite element method (SBFEM) and precise integration methodology (PIM). Locations and number of supplemental concentrated masses are not restricted in the proposed approach. Only the length of the beam model is needful to be discretized with the help of spectral elements. Based on the scaled boundary coordinate system, partial differential equations of the elastic beam are transformed into the second order ordinary differential matrix equation. By virtue of the dual vector, it is convenient to obtain a further simplified first order ordinary differential equation, which is solved by PIM to acquire the stiffness matrix. In light of coupling the same degrees of freedom, the global mass matrix is gained. Calculating the eigenvalue equation brings free vibration frequencies of the elastic beam with extra point masses. Comparisons with available results provided by literatures are presented to reveal the high accuracy of the introduced SBFEM.

1. Introduction

Elastic beams are widely used in engineering structures. To ensure the structural safety and performance of beams, it is necessary to explore vibration characteristics and solve natural frequencies. In practice, beam structures often carry attached concentrated masses, which significantly alter flexural frequencies. In order to accurately predict dynamic responses, optimize design and prevent resonance-related failures, investigations on variations of eigenfrequencies for elastic homogeneous beams with added point masses are essential.

In recent years, several studies have focused on the impact of attached masses on vibration responses of elastic beams. Li et al.[1] analyzed transverse vibration behaviors

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of a cantilever beam under the axial force and tip mass and proposed a new integral equation method for more accurately predicting flexural frequencies. Torabi et al.[2] proposed an exact closed-form solution for vibration modes of a Timoshenko beam with multiple concentrated masses and revealed that added masses significantly reduced natural frequencies of the beam. Shi et al.[3] studied the impact of unequal end masses on vibration frequencies of a free-free beam and introduced the Fredholm integral equation to approximate resonant frequencies. Aksencer and Aydogdu[4] used the Ritz method to analyze free vibration behaviors of rotating composite beams and demonstrate the effect of attached mass on flexural frequencies. Rahmani et al.[5] employed the modified couple stress theory and Rayleigh-Ritz method to explore the effect of attached masses on distributions of natural frequencies for micro-beams. These studies highlight that positions and magnitudes of attached masses play a significant role in the vibration characteristics of beams.

The structure of this paper is organized as follows. Section 2 briefly outlines the solution procedure to transverse vibration frequencies by the SBFEM. In Section 3, three numerical examples are presented to validate the proposed method and highlight the effectiveness. Section 4 concludes the study with a summary of key results.

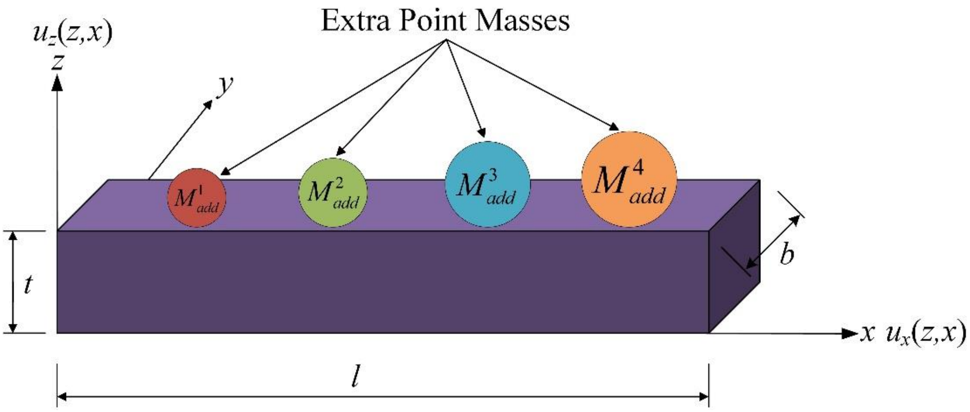

Fig. 1 The elastic beam with four point masses.

2. Solution procedure

This section presents the derivation and solution process of governing equations for elastic homogeneous beams carrying extra concentrated masses using the SBFEM coupled with the PIM. Fig. 1 displays that the elastic beam with the length *l*, width *b* and thickness *t* carries four point masses M^i_{add} (i=1, 2, 3, 4).

In the plane-stress state, the stress-strain relation of the beam is formulated as

$$\{\sigma\} = [C]\{\varepsilon\} \tag{2.1}$$

$$\{\sigma\} = [\sigma_{zz} \quad \sigma_{xx} \quad \tau_{xz}]^T \tag{2.2}$$

$$\{\varepsilon\} = [\varepsilon_{zz} \quad \varepsilon_{xx} \quad \gamma_{xz}]^T \tag{2.3}$$

where the constitutive matrix in Eq. (2.1) is denoted as

$$[C]=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & \\ \nu & 1 & \\ & & (1-\nu)/2\end{bmatrix} \tag{2.4}$$

Meanwhile, $\{\sigma\}$ and $\{\varepsilon\}$ in Eqs. (2.2) and (2.3) stand for vectors of the stress and strain. In Eq. (2.4), E and ν mean elastic modules and Poisson's ratio of the beam structure.

The beam is discretized along its longitudinal axis using high-order spectral elements. Each node of the spectral element has two degrees of freedom: elastic displacements $u_z(z,x)$ and $u_x(z,x)$. A scaled boundary coordinate system (z,η) is applied to simplify the solution procedure.

The *x*-coordinate and displacement field are interpolated as

$$x(\eta)=\left[N(\eta)\right]\{x\}=[N]\{x\} \tag{2.5}$$

$$\{u(z,\eta)\}=\begin{bmatrix}\left[N(\eta)\right] & \\ & \left[N(\eta)\right]\end{bmatrix}\begin{bmatrix}\{u_z(z)\}\\ \{u_x(z)\}\end{bmatrix}=[\mathbf{N}]\{u(z)\} \tag{2.6}$$

where $\{x\}$ is constituted by nodal coordinates of the spectral element and the shape function matrix $\left[N(\eta)\right]$ is expressed as the Lagrange polynomial.

With the help of the strain-displacement relationship, the strain vector is rewritten as

$$\{\varepsilon\}=\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}[\mathbf{N}]\frac{d\{u(z)\}}{dz}+\frac{1}{|J|}\begin{bmatrix}0 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}\frac{d[\mathbf{N}]}{d\eta}\{u(z)\}=\left[B^1\right]\{u(z)\}_{,z}+\left[B^2\right]\{u(z)\} \tag{2.7}$$

with $\left[B^1\right]$ and $\left[B^2\right]$ defined as

$$\left[B^1\right]=\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}[\mathbf{N}]\quad \left[B^2\right]=\frac{1}{|J|}\begin{bmatrix}0 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}\frac{d[\mathbf{N}]}{d\eta}=\frac{1}{|J|}\begin{bmatrix}0 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}[\mathbf{N}]_{,\eta} \tag{2.8}$$

In Eq. (2.7), $|J|=\left[N(\eta)\right]_{,\eta}\{x\}$ represents the Jacobian determinant.

The stress field is formulated as

$$\{\sigma\}=[C](\left[B^1\right]\{u(z)\}_{,z}+\left[B^2\right]\{u(z)\}) \tag{2.9}$$

Applying the principle of virtual work yields the second-order ordinary differential equation

$$[E^0]\{u(z)\}_{,zz}+([E^1]^T-[E^1])\{u(z)\}_{,z}-[E^2]\{u(z)\}=0 \tag{2.10}$$

where constant coefficient matrices are denoted as

$$[E^0]=\int_{-1}^{1}[B^1]^T[C][B^1]|J|bd\eta \tag{2.11}$$

$$[E^1]=\int_{-1}^{1}[B^2]^T[C][B^1]|J|bd\eta \tag{2.12}$$

$$[E^2]=\int_{-1}^{1}[B^2]^T[C][B^2]|J|bd\eta \tag{2.13}$$

Introducing the nodal force vector

$$\{q(z)\} = [E^0]\{u(z)\}_{,z} + [E^1]^T\{u(z)\} \tag{2.14}$$

forms the variable $\{X(z)\}$

$$\{X(z)\} = \begin{Bmatrix} \{u(z)\} \\ \{q(z)\} \end{Bmatrix} \tag{2.15}$$

By virtue of Eq. (2.15), Eq. (2.10) is rewritten as

$$\{X(z)\}_{,z} = -[Z]\{X(z)\} \tag{2.16}$$

with the coefficient matrix

$$[Z] = \begin{bmatrix} [E^0]^{-1}[E^1]^T & -[E^0]^{-1} \\ -[E^2]+[E^1][E^0]^{-1}[E^1]^T & -[E^1][E^0]^{-1} \end{bmatrix} \tag{2.17}$$

The solution to Eq. (2.16) is obtained

$$\{X(z)\} = e^{-[Z]z}\{c\} \tag{2.18}$$

where $e^{-[Z]z}$ stands for the matrix exponential.

Adopting the PIM with the thickness division, the stiffness matrix $[K]$ related to displacements and external forces is derived

$$[K]\begin{Bmatrix} \{u_B\} \\ \{u_T\} \end{Bmatrix} = \begin{Bmatrix} \{F_B\} \\ \{F_T\} \end{Bmatrix} \tag{2.19}$$

In Eq. (2.19), $\{u_B\}$, $\{F_B\}$ and $\{u_T\}$, $\{F_T\}$ symbolize displacements and external forces at top (*z*=*t*) and bottom (*z*=0) surfaces, respectively.

Aided by the kinetic energy, the consistent mass matrix for the homogeneous beam is expressed as

$$[M] = \begin{bmatrix} M_B & 0 \\ 0 & M_T \end{bmatrix} \tag{2.20}$$

Submatrices M_T and M_B are identical due to the symmetry

$$M_T = M_B = \frac{t}{2}\int_{-1}^{1}[\mathbf{N}]^T \rho[\mathbf{N}]b|J|d\eta \tag{2.21}$$

where ρ is the density of the beam structure.

Added point masses M_{add}^i at nodes and the diagonal entries of [M] are integrated according to the rule of matching same degrees of freedom (DOFs). For k concentrated masses at DOFs d_1, d_2, …, d_k, the global mass matrix is denoted as

$$[M_c]_{ii} = \begin{cases} [M]_{ii} + M_{add}^i & if \quad i = d_i \\ [M]_{ii} & \text{otherwise} \end{cases} \tag{2.22}$$

Natural frequencies ω to the elastic beam with extra point masses are calculated by

$$\left\|[K] - \omega^2[M_c]\right\| = 0 \tag{2.23}$$

3. Numerical examples

Built upon the above derivation, vibration frequencies of elastic beams carrying concentrated masses with different positions and numbers are solved in this section. To illustrate the accuracy of the developed technology, comparisons with reference results in related literatures are provided. The dimensionless formulas for mass ratios and positions of additional point masses as well as natural frequencies are expressed as

$$\alpha_i = \frac{M^i_{add}}{\rho A l} \tag{2.24}$$

$$\eta_i = \frac{x_i}{l} \tag{2.25}$$

$$\omega_0 = \omega l^2 \sqrt{\rho A / EI} \tag{2.26}$$

where A and I are the area and moment of inertia respectively.

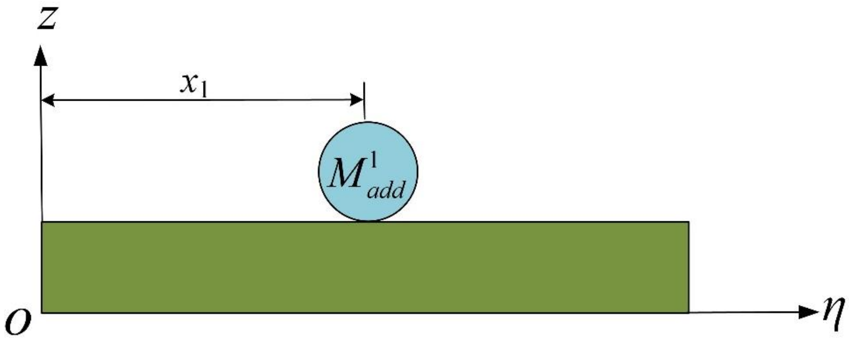

Fig. 2 The elastic beam with a single point mass.

3.1. The elastic beam with a single point mass

In this section, free vibration behaviors of the elastic beam with a single point mass are investigated, as illustrated in Fig. 2. Dimensions of the beam are *l*=1, *b*=0.03 and *t*=0.01. The point mass with $\alpha = 1$ is placed at various positions η=0.0, 0.1, 0.2, 0.3, 0.4, 0.5 and 0.6. Three boundary conditions are considered: simple-simple, clamped-simple and clamped-clamped. Solutions to natural frequencies under different constraints are displayed in Tables 1-3. Results obtained by the proposed SBFEM are compared with reference solutions from Ref. [6]. Tables 1–3 show that computed vibration frequencies agree closely with existing results with relative errors generally below 0.7%. As a result, the accuracy and effectiveness of the introduced approach are validated.

Table 1 Vibration frequencies of the beam with one point mass under simple-simple supports

		ω_1	ω_2	ω_3	ω_4	ω_5
η=0.0	Ref. [6]	9.8695	39.4784	88.8264	157.9144	246.7413
	SBFEM	9.8675	39.4441	88.6529	157.3669	245.4107
	Error(%)	-0.0207	-0.0870	-0.1953	-0.3467	-0.5393
η=0.1	Ref. [6]	8.9962	29.8891	66.0691	127.2135	213.3439
	SBFEM	8.9943	29.8653	65.9558	126.8271	212.7481
	Error(%)	-0.0206	-0.0796	-0.1715	-0.3037	-0.2793
η=0.2	Ref. [6]	7.4541	26.9462	73.5140	149.3992	246.7413
	SBFEM	7.4528	26.9277	73.3878	148.8992	245.6657
	Error(%)	-0.0178	-0.0686	-0.1717	-0.3347	-0.4359
η=0.3	Ref. [6]	6.3946	29.7503	86.7293	143.2258	209.3172
	SBFEM	6.3937	29.7291	86.5623	142.7514	208.3562
	Error(%)	-0.0146	-0.0713	-0.1926	-0.3312	-0.4591
η=0.4	Ref. [6]	5.8468	35.2374	79.9788	132.6574	246.7413

	SBFEM	5.8460	35.2090	79.8315	132.2470	245.4106
	Error(%)	-0.0141	-0.0806	-0.1842	-0.3094	-0.5393
η=0.5	Ref. [6]	5.6795	39.4784	67.8883	157.9144	206.7901
	SBFEM	5.6788	39.4441	67.7764	157.3669	205.7803
	Error(%)	-0.0127	-0.0870	-0.1648	-0.3467	-0.4883

Table 2 Natural frequencies of the beam with one point mass under clamped-simple supports

		ω_1	ω_2	ω_3	ω_4	ω_5
η=0.1	Ref. [6]	22.3733	61.6728	120.9032	199.8604	298.5569
	SBFEM	22.3593	61.5732	120.5395	198.8947	296.4504
	Error(%)	-0.0628	-0.1614	-0.3008	-0.4832	-0.7055
η=0.2	Ref. [6]	21.9474	53.8427	89.8598	151.9623	243.0824
	SBFEM	21.9325	53.7195	89.5176	151.2708	242.8319
	Error(%)	-0.0677	-0.2288	-0.3809	-0.4550	-0.1031
η=0.3	Ref. [6]	18.3360	40.9434	93.3305	177.8542	290.1980
	SBFEM	18.3220	40.8814	93.0922	177.0728	289.1711
	Error(%)	-0.0764	-0.1515	-0.2553	-0.4393	-0.3539
η=0.4	Ref. [6]	14.4030	44.2995	112.5615	195.4739	254.3674
	SBFEM	14.3939	44.2416	112.2408	194.5278	252.7315
	Error(%)	-0.0631	-0.1307	-0.2849	-0.4840	-0.6431
η=0.5	Ref. [6]	12.4047	53.5218	114.5992	167.6507	297.2762
	SBFEM	12.3982	53.4440	114.2643	166.9303	295.1939
	Error(%)	-0.0524	-0.1454	-0.2922	-0.4297	-0.7005
η=0.6	Ref. [6]	11.8182	61.6727	95.7568	199.8604	253.7298
	SBFEM	11.8124	61.5732	95.5118	198.8947	252.0927
	Error(%)	-0.0494	-0.1613	-0.2559	-0.4832	-0.6452

Table 3 Eigenfrequencies of the beam with one point mass under clamped-clamped supports

		ω_1	ω_2	ω_3	ω_4	ω_5
η=0.0	Ref. [6]	15.2752	45.5767	79.3377	133.4672	217.8576
	SBFEM	15.2687	45.5007	79.0580	132.9238	217.1437
	Error(%)	-0.0426	-0.1668	-0.3526	-0.4071	-0.3277
η=0.1	Ref. [6]	13.8203	33.2808	77.0176	153.7460	259.8318
	SBFEM	13.8135	33.2373	76.8597	153.3875	258.7435
	Error(%)	-0.0490	-0.1306	-0.2051	-0.2332	-0.4188
η=0.2	Ref. [6]	11.3683	33.0378	92.2403	178.0890	234.5798
	SBFEM	11.3631	33.0052	92.0381	177.3521	233.2119
	Error(%)	-0.0460	-0.0986	-0.2192	-0.4138	-0.5831
η=0.3	Ref. [6]	9.6093	38.6505	103.6283	145.8877	263.2084
	SBFEM	9.6056	38.6119	103.3719	145.3552	261.6470
	Error(%)	-0.0381	-0.0999	-0.2474	-0.3650	-0.5932
η=0.4	Ref. [6]	8.6977	47.2840	84.6891	172.7437	236.1355
	SBFEM	8.6949	47.2299	84.5039	172.0606	234.7557
	Error(%)	-0.0327	-0.1143	-0.2187	-0.3954	-0.5843
η=0.5	Ref. [6]	8.4780	48.5385	87.0356	158.8255	266.7995
	SBFEM	8.4724	48.4798	86.8500	158.2016	265.1883
	Error(%)	-0.0662	-0.1210	-0.2132	-0.3928	-0.6039

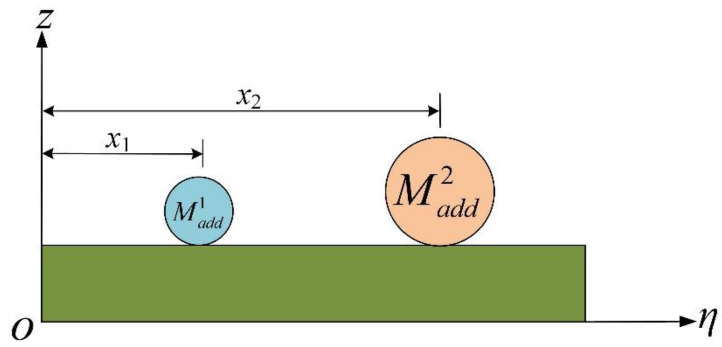

Fig. 3 The homogeneous beam carrying two attached concentrated masses.

3.2. The homogeneous beam carrying two attached concentrated masses

This section explores free vibration responses of the homogeneous beam carrying two attached concentrated masses, as portrayed in Fig. 3. The geometric dimensions are the same as that in Section 3.1. Two point masses are located at positions η_1=0.1 η_2=0.4 and η_1=0.5 η_2=0.7 with mass ratios α_1=1 α_2=1, α_1=1 α_2=10, α_1=10 α_2=1 and α_1=10 α_2=10. Table 4 presents natural frequencies of the beam with simple-simple boundary constraints, where two masses are located at η_1=0.5 and η_2=0.7. In the Table 5, eigenfrequencies for the clamped-free beam with additional masses at η_1=0.1 and η_2=0.4 are provided. In all cases, solutions computed by the employed SBFEM show excellent agreement with reference results from Ref. [7].

Table 4 Frequency parameters of the beam with two point masses under simple-simple conditions

			ω_1	ω_2	ω_3	ω_4
η_1=0.5 η_2=0.7	α_1=1 α_2=1	Ref. [7]	4.7297	25.1279	60.8832	141.2890
		SBFEM	4.7299	25.1099	60.7864	140.8179
		Error(%)	0.0049	-0.0717	-0.1590	-0.3334
	α_1=1 α_2=10	Ref. [7]	2.3875	17.9251	59.5695	136.9930
		SBFEM	2.3871	17.9125	59.4754	136.5352
		Error(%)	-0.0154	-0.0702	-0.1579	-0.3342
	α_1=10 α_2=1	Ref. [7]	2.0777	22.0363	54.6468	140.866
		SBFEM	2.0775	22.0198	54.5630	140.3942
		Error(%)	-0.0101	-0.0750	-0.1533	-0.3349
	α_1=10 α_2=10	Ref. [7]	1.6769	9.8120	53.5165	136.5350
		SBFEM	1.6768	9.8045	53.4360	136.0769
		Error(%)	-0.0079	-0.0767	-0.1505	-0.3355

Table 5 Flexural frequencies of the beam with two lumped masses under clamped-free conditions

			ω_1	ω_2	ω_3	ω_4
η_1=0.1 η_2=0.4	α_1=1 α_2=1	Ref. [7]	3.1802	13.5261	50.8105	74.6163
		SBFEM	3.1798	13.5184	50.7160	74.3114
		Error(%)	-0.0118	-0.0569	-0.1860	-0.4086
	α_1=1 α_2=10	Ref. [7]	1.8816	8.4921	49.5416	71.4882
		SBFEM	1.8814	8.4881	49.4555	71.1900
		Error(%)	-0.0124	-0.0466	-0.1738	-0.4171
	α_1=10 α_2=1	Ref. [7]	3.1645	12.6406	26.0392	56.8499
		SBFEM	3.1641	12.6293	25.9276	56.7305
		Error(%)	-0.0127	-0.0898	-0.4286	-0.2100
	α_1=10 α_2=10	Ref. [7]	1.8773	8.4071	24.6129	53.0191
		SBFEM	1.8770	8.4028	24.4975	52.9161
		Error(%)	-0.0151	-0.0516	-0.4687	-0.1942

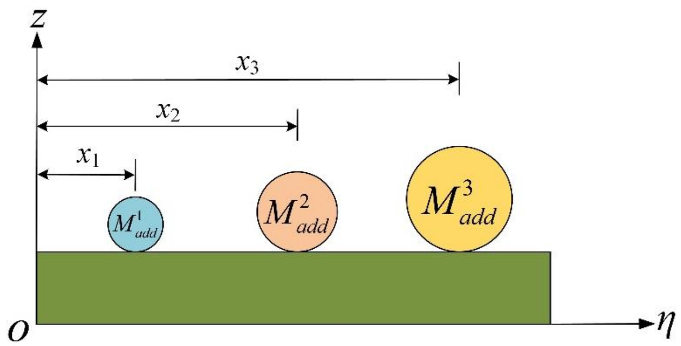

Fig. 4 The beam structure with three added lumped masses.

3.3. The beam structure with three added lumped masses

This section examines changing rules of vibration frequencies for the beam structure with three added lumped masses, as demonstrated in Fig. 4. The identical dimensional parameters with Section 3.1 are adopted. Two types of boundary conditions are under consideration: clamped-clamped and simple-clamped. Positions of three point masses are η_1=0.1, η_2 =0.4, η_3=0.8 and η_1=0.2, η_2=0.5, η_3=0.7. Five kinds of mass ratios α_1, α_2 and α_3 are discussed. Natural frequencies of elastic beams with different constrain conditions and extra masses are exhibited in Tables 6-7. In all test situations, eigenfrequencies provided by the utilized SBFEM show strong agreement with reference values from Ref. [7] with relative errors less than 0.5%, which further confirms the accuracy, stability and applicability of the present methodology for solving free vibration problems of elastic beams carrying multiple point masses under various boundary conditions.

Table 6 Natural frequencies of the beam with three added masses and clamped-clamped supports

			ω_1	ω_2	ω_3	ω_4
	α_1=1	Ref. [7]	11.7922	30.7215	67.7822	110.9670
	α_2=1	SBFEM	11.7856	30.6750	67.5185	110.5599
	α_3=1	Error(%)	-0.0556	-0.1513	-0.3890	-0.3669
	α_1=1	Ref. [7]	7.6383	17.1981	66.3658	105.8490
	α_2=1	SBFEM	7.6310	17.1781	66.1164	105.4602
	α_3=10	Error(%)	-0.0952	-0.1164	-0.3758	-0.3673
η_1=0.1	α_1=1	Ref. [7]	4.5411	28.6699	66.8277	107.8630
η_2=0.4	α_2=10	SBFEM	4.5387	28.6277	66.5641	107.4802
η_3=0.8	α_3=1	Error(%)	-0.0522	-0.1473	-0.3945	-0.3549
	α_1=10	Ref. [7]	11.1358	24.0862	34.1821	105.6040
	α_2=1	SBFEM	11.1270	23.9996	34.0930	105.2615
	α_3=1	Error(%)	-0.0786	-0.3595	-0.2607	-0.3244
	α_1=10	Ref. [7]	4.2900	10.8779	25.0046	97.4892
	α_2=10	SBFEM	4.2877	10.8612	24.8826	97.1895
	α_3=10	Error(%)	-0.0544	-0.1532	-0.4878	-0.3074

Table 7 Vibration frequencies of the beam with three added masses and simple-clamped supports

			ω_1	ω_2	ω_3	ω_4
	α_1=1	Ref. [7]	6.9686	21.2964	43.3097	161.7880
	α_2=1	SBFEM	6.9665	21.2790	43.2375	161.1664
η_1=0.2	α_3=1	Error(%)	-0.0308	-0.0815	-0.1667	-0.3842
η_2=0.5	α_1=1	Ref. [7]	4.2861	14.3862	35.9369	161.6720
η_3=0.7	α_2=1	SBFEM	4.2840	14.3754	35.8829	161.0509
	α_3=10	Error(%)	-0.0479	-0.0751	-0.1502	-0.3842
	α_1=1	Ref. [7]	3.1284	20.6621	34.4942	159.3900

α_2=10	SBFEM	3.1275	20.6465	34.4366	158.7790
α_3=1	Error(%)	-0.0296	-0.0755	-0.1671	-0.3833
α_1=10	Ref. [7]	3.8659	14.1970	40.5099	159.7870
α_2=1	SBFEM	3.8648	14.1862	40.4429	159.1712
α_3=1	Error(%)	-0.0276	-0.0761	-0.1655	-0.3854
α_1=10	Ref. [7]	2.4381	7.3628	14.8312	156.9830
α_2=10	SBFEM	2.4374	7.3575	14.8076	156.3815
α_3=10	Error(%)	-0.0295	-0.0725	-0.1594	-0.3832

4. Conclusion

This paper presented the numerical study on free vibration behaviors of elastic beams with additional point masses based on the SBFEM and PIM. The effectiveness of the proposed approach is confirmed through several numerical examples involving different point masses. The computed natural frequencies show excellent agreement with benchmark results from existing literatures, which releases both the reliability and accuracy of the introduced procedure.

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